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Affine structures on 4-step nilpotent lie algebras

Karel Dekimpe[a,1,*], Manfred Hartl[b,2]

[a] *Katholieke Universiteit Leuven, Campus Kortrijk, B-8500 Kortrijk, Belgium*
[b] *Institut des Sciences et Techniques de Valenciennes, Université de Valenciennes, B.P. 311, F-59304 Valenciennes, France*

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Abstract

In this paper we construct complete left-symmetric (and so affine) structures on a class of 4-step nilpotent Lie algebras. To achieve this, we use the language of derivations and translate the problem of the existence of a complete left-symmetric structure for a given Lie algebra L, to the existence of a certain submodule (called layerwise complementary) of the augmentation ideal of the universal enveloping algebra of L. A polynomial construction (an analogue of the classical polynomial construction for groups), due to the second author, is used to determine such a submodule for all 3-step nilpotent Lie algebras (allowing to rediscover the known results) and for a reasonable class of 4-step nilpotent Lie algebras (for which the existence of a complete left-symmetric/affine structure was not known before). A concrete description of this left-symmetric structure is given. © 1997 Elsevier Science B.V.

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1. Introduction

In 1977, Milnor asked whether each torsion-free polycyclic-by-finite group can be realized as the fundamental group of a compact, complete and affinely flat manifold [11]. For nilpotent groups of class 3, a positive answer was already provided in 1974, by the work of Scheuneman [14].

* Corresponding author.

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Since then, no real progress has been made until recently when a negative answer to Milnor's question was given, even in the nilpotent case, by Benoist [1, 2] and Burde and Grunewald [4]. The counterexamples they provide are of rank 11 and nilpotency class 10. These examples have a strictly positive affine defect number (equal to 1) [6]. On the other hand, the question is still open even for most groups of nilpotency class 4.

In this paper, we attack, as Scheuneman did, the problem on the Lie algebra level. This means that for a given Lie algebra L, we try to establish a faithful affine representation $L \to \mathfrak{aff}(\mathbb{R}^m)$ which is minimal, i.e. where the dimension m coincides with that of L.

To achieve this, we introduce a new method for constructing minimal affine representations by using certain polynomial constructions related to those of Passi on the group level [12]. Not only some important known results are reproduced from this unifying point of view but also a substantial class of 4-step nilpotent Lie algebras is covered for which the existence problem was not solved so far, e.g. all 4-step nilpotent Lie algebras generated by 2 elements belong to this family. The statement of our main theorem is as follows:

Theorem A. *Let L be a nilpotent Lie algebra. Suppose L has a 4-step filtration $L = \mathscr{L}_1 \supseteq \mathscr{L}_2 \supseteq \mathscr{L}_3 \supseteq \mathscr{L}_4 \supseteq \mathscr{L}_5 = 0$ satisfying $[\mathscr{L}_i, \mathscr{L}_j] \subseteq \mathscr{L}_{i+j}$, such that the natural (bracket) map from $\Lambda^2(\mathscr{L}_1/\mathscr{L}_2) \to \mathscr{L}_2/\mathscr{L}_3$ is injective. Then there is a 4-step filtered L-module $M = M_1 \supseteq M_2 \supseteq M_3 \supseteq M_4 \supseteq M_5 = 0$ satisfying $\mathscr{L}_i \cdot M_j \subseteq M_{i+j}$, and a derivation t which maps $\mathscr{L}_i$ bijectively onto M_i for each i. Consequently, L admits a complete left-symmetric structure and so also a minimal faithful affine representation.*

Note that a 4-step nilpotent Lie algebra L, for which $L/[L,[L,L]]$ is free nilpotent of class 2, satisfies the hypothesis of the theorem, if the filtration is defined inductively by $\mathscr{L}_1 = L$ and $\mathscr{L}_{i+1} = [L, \mathscr{L}_i]$. This includes the case that L is generated by two elements.

Another way of formulating this result is as follows:

Theorem B. *Let L be a 4-step nilpotent Lie algebra containing a subspace U, with $[L,L] \subseteq U \subseteq L$ such that*

(1) $[L,[L,[L,U]]] = [U,[U,L]] = 0$ *and*

(2) $\dim([L,L]/[U,L]) = \binom{\dim(L/U)}{2}$.

Then L admits a complete left-symmetric structure and so also a minimal faithful affine representation.

We would also like to formulate a concrete way to build up 4-step nilpotent Lie algebras satisfying the conditions of Theorem A.

Theorem C. *Assume $X = \{x_1, x_2, \ldots, x_m\}$ is a minimal set of generators for a 4-step nilpotent Lie algebra L over a field k of characteristic 0. Let $\{R_i = \sum_k c_{i_k}[x_{i_k}, y_{j_k}]\}$ be a basis of*

$$\operatorname{Ker}\left(\bigoplus_{1\le p<q\le m} k[x_p,x_q] \twoheadrightarrow [L,L]/[L,[L,L]]\right).$$

Suppose $\exists S \subseteq X$ such that
(1) $\forall i_k : c_{i_k} \neq 0 \Rightarrow x_{i_k} \in S$ *or* $y_{i_k} \in S$.
(2) $[X,[X,[X,S]]] = 0 = [S,[S,X]]$.
Then L admits a minimal faithful affine representation.

Here are some examples to which the theorem applies:
(1) $\{R_i\} = \emptyset$ (with $S = \emptyset$).
(2) Suppose there exists a subset $T \subseteq X$ such that all R_i are of the form

$$R_i = [t,x], \quad \text{with } t \in T,\ x \in X.$$

If moreover $[X,[X,[X,T]]] = [T,[T,X]] = 0$, we can apply Theorem B (with $S = T$). For example, the latter condition is automatically satisfied if $\{R_i\} = \{[t,x] \parallel t \in T, x \in X\}$.
These examples allow to write down immediately presentations of many Lie algebras satisfying the hypothesis of Theorem A.

2. Affine representations and nonsingular derivations

First we point out how Milnor's problem for nilpotent groups can be translated completely into the language of derivations of nilpotent Lie algebras. By [7], any compact complete affinely flat manifold with nilpotent fundamental group is an affine nilmanifold, i.e. a quotient space $\Gamma\backslash G$, where G is a simply connected nilpotent Lie group having a complete left-invariant flat affine connection and Γ is a uniform lattice of G.

In [10] Hyuk Kim proves the following proposition:

Proposition 2.1. *Let G be a simply connected Lie group with its Lie algebra L. Then the following are equivalent:*
(1) *G admits a complete left-invariant flat affine connection.*
(2) *G acts simply transitively on a vector space M by affine transformations.*
(3) *There exists a complete left-symmetric structure on L.*

Recall that a left-symmetric structure on L is given by a Lie algebra homomorphism $\lambda : L \to gl(L)$ in such a way that, with respect to the module structure of L determined by λ, the identity map $1_L : L \to L$ is a derivation. For convenience of the reader, we recall the definition of the latter notion.

Definition 2.2. Let L be a Lie algebra over a field k and M a left L-module. A k-linear map $t : L \to M$ is called a derivation if

$$\forall x, y \in L:\ t[x, y] = x \cdot t(y) - y \cdot t(x).$$

Continuing our discussion on left-symmetric structures, we point out that in case L is a nilpotent Lie algebra, it follows also from the work of Kim that a left-symmetric structure is complete if and only if λ is a nilpotent representation. As a conclusion, we find:

Theorem 2.3. *A connected and simply connected nilpotent Lie group G admits a complete left-invariant flat affine connection if and only if the corresponding Lie algebra L admits a faithful derivation into a nilpotent L-module M, which has the same dimension as L.*

This leads to the following definition:

Definition 2.4. Let L be any Lie algebra over a field k and M a left L-module. A derivation $t : L \to M$ is called nonsingular if t is a faithful linear map. t is called minimal if $\dim_k L = \dim_k M$.

Another way of looking at derivations is to consider them as being the translational part of an affine representation. More precisely, if k is any field, we let

$$\mathfrak{aff}(m) = \left\{ \begin{pmatrix} A & a \\ 0 & 0 \end{pmatrix} \| A \in \mathrm{gl}(m, k),\ a \in k^m \right\}.$$

$\mathfrak{aff}(m)$ is made into a Lie algebra over k in the usual way and is called the affine m-dimensional Lie algebra. Let us consider the following two maps, called the linear part and the translational part, respectively:

$$\mathrm{lin} : \mathfrak{aff}(m) \to \mathrm{gl}(m, k) : \begin{pmatrix} A & a \\ 0 & 0 \end{pmatrix} \mapsto A$$

and

$$\mathrm{tr} : \mathfrak{aff}(m) \to k^m : \begin{pmatrix} A & a \\ 0 & 0 \end{pmatrix} \mapsto a.$$

Let L be a Lie algebra over k and suppose there exists a representation $\rho : L \to \mathfrak{aff}(m)$, then it is obvious that $\mathrm{tr}(\rho) : L \to k^m$ is a derivation, where the module structure of k^m is given by $\mathrm{lin}(\rho)$. Conversely, if $t : L \to k^m$ is a derivation into the L-module k^m

and if this module structure is given by a map $\lambda : L \to \mathrm{gl}(m,k)$, there is an affine representation

$$\rho : L \to \mathfrak{aff}(m) : x \mapsto \begin{pmatrix} \lambda(x) & t(x) \\ 0 & 0 \end{pmatrix}.$$

Therefore, in case $k = \mathbb{R}$, the statement of Theorem 2.3 is also equivalent to the existence of a Lie algebra homomorphism $L \to \mathfrak{aff}(m)$, for which the linear parts are nilpotent and the translational part determines a vector space isomorphism $L \to k^m$.

We shall make use of the fact that nilpotent Lie algebras admit nice filtrations of finite length. Therefore we shall consider derivations/affine representations which are related to such filtrations.

Definition 2.5. Let L be a Lie algebra over a field k of characteristic 0. By an N-series of L we shall mean any descending filtration by subspaces:

$$\mathscr{L} : L = \mathscr{L}_1 \supseteq \mathscr{L}_2 \supseteq \mathscr{L}_3 \supseteq \mathscr{L}_4 \supseteq \cdots$$

which satisfies $[\mathscr{L}_i, \mathscr{L}_j] \subseteq \mathscr{L}_{i+j}$ for any $i,j \geq 1$.

Note that the latter condition implies that all spaces $\mathscr{L}_i$ are actually ideals in L.

A well-known example of such an N-series is the lower central series of L. Here $\mathscr{L}_1 = L = \gamma_1(L)$ and $\mathscr{L}_{i+1} = [L, \mathscr{L}_i] = [L, \gamma_i(L)] = \gamma_{i+1}(L)$ for $i \geq 1$. We shall refer to the lower central series by the notation $\mathscr{L} = \gamma$. Associated to any N–series $\mathscr{L}$ of L, there is a graded Lie algebra $\mathrm{Gr}^{\mathscr{L}}(L)$, with $\mathrm{Gr}_i^{\mathscr{L}}(L) = \mathscr{L}_i / \mathscr{L}_{i+1}$.

Notation 2.6. When no confusion can arise, we shall write L_i instead of $\mathrm{Gr}_i^{\mathscr{L}}(L)$.

We shall write $k_i = \dim_k \mathrm{Gr}_i^{\mathscr{L}}(L)$. If $\mathscr{L}_n \neq 0$ and $\mathscr{L}_{n+1} = 0$, the N-series is said to be of length n.

In [5] the definition of a canonical form affine representation of a nilpotent Lie algebra was given. This is a faithful affine representation which respects, in some sense, the upper central series. However, the definition can easily be extended to more general central series of a given Lie algebra. Here, we consider a subclass of these representations:

Definition 2.7. A faithful affine representation $\rho : L \to \mathfrak{aff}(m)$ will be called of strong canonical form with respect to an N-series $\mathscr{L}$ of L of length n if and only if there exist subspaces $M_i \subseteq k^m$ $(1 \leq i \leq c)$, with $M_1 = k^m$, such that

(1) $\forall i \in \{1,2,\ldots,n\}$: $\mathrm{tr}(\rho) : \mathscr{L}_i \to M_i$ is an isomorphism of vector spaces.
(2) $\forall i,j \in \{1,2,\ldots,n\}$: $\mathrm{lin}(\rho)(\mathscr{L}_i) \cdot M_j \subseteq M_{i+j}$, where $M_p = 0$ if $p > n$.

Remark 2.8. An affine representation $\rho : L \to \mathfrak{aff}(m)$ of an m-dimensional Lie algebra is of strong canonical form iff there exists a basis of k^m for which the matrix expression of ρ looks as follows:

$$\forall x \in \mathscr{L}_j :\ \rho(x) = \begin{pmatrix} O_{k_n} & \cdots & \cdots & \cdots & \cdots & & & * & * & * \\ \vdots & & & & & & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & O_{k_{j+2}} & 0 & 0 & 0 & \ddots & * & * & * \\ 0 & \cdots & 0 & O_{k_{j+1}} & 0 & 0 & \ddots & 0 & * & * \\ 0 & \cdots & 0 & 0 & O_{k_j} & 0 & \ddots & 0 & 0 & * \\ 0 & \cdots & 0 & 0 & 0 & O_{k_{j-1}} & \ddots & 0 & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & O_{k_1} & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}$$

such that the translational parts form all of k^m.

In this notation, O_i is used to indicate an $i \times i$ zero matrix, so all other zeroes appearing above also represent zero blocks.

To see examples of such affine representations the reader should look at those obtained by Scheuneman ([14], cf. also Section 4 below), which are of strong canonical form with respect to the lower central series.

Now we want to determine a class of derivations which behave nicely with respect to a given N-series $\mathscr{L}$. To do so, we need a filtration of the universal enveloping algebra $\mathscr{U}(L)$ of L.

Definition 2.9.

$$F_0^{\mathscr{L}}(L) = \mathscr{U}(L) \quad \text{and} \quad \forall i > 0 :\ F_i^{\mathscr{L}}(L) = \sum_{\substack{s \\ i_1+i_2+\cdots+i_s \geq i}} \mathscr{L}_{i_1}\mathscr{L}_{i_2}\ldots\mathscr{L}_{i_s} \subseteq \mathscr{U}(L),$$

where the $\mathscr{L}_i$ are seen as subspaces of $\mathscr{U}(L)$, via the canonical embedding of L into $\mathscr{U}(L)$.

It is clear that the $F_i^{\mathscr{L}}(L)$ determine a filtration of $\mathscr{U}(L)$ by two-sided ideals and that $F_1^{\mathscr{L}}(L) = \overline{\mathscr{U}}(L)$, the augmentation ideal of $\mathscr{U}(L)$.

Now suppose that $t : L \to M$ is a derivation of L into an L-module M. This derivation, together with the module structure of M, determines a filtration of M, where the ith term of the filtration is given by

$$\forall i \geq 1 :\ F_{t,i}^{\mathscr{L}}(M) = \sum_{0 \leq j \leq i-1} F_j^{\mathscr{L}}(L) \cdot t(\mathscr{L}_{i-j}) + F_{i-1}^{\mathscr{L}}(L) \cdot M.$$

Note that $F_{t,i}^{\mathscr{L}}(M)$ is an L-submodule of M and that $\mathscr{L}_k \cdot F_{t,i}^{\mathscr{L}}(M) \subseteq F_{t,k+i}^{\mathscr{L}}(M)$. Thus the graded k-space $\mathrm{Gr}^{\mathscr{L},t}(M)$ with $\mathrm{Gr}_i^{\mathscr{L},t}(M) = F_{t,i}^{\mathscr{L}}(M)/F_{t,i+1}^{\mathscr{L}}(M)$ is a module over the graded Lie algebra $\mathrm{Gr}^{\mathscr{L}}(L)$. Moreover, a $\mathrm{Gr}^{\mathscr{L}}(L)$-derivation $\mathrm{Gr}^{\mathscr{L}}(t) : \mathrm{Gr}^{\mathscr{L}}(L) \to \mathrm{Gr}^{\mathscr{L},t}(M)$ is well defined by t since $t(\mathscr{L}_i) \subseteq F^{\mathscr{L}}_{t,i}(M)$.

Notation 2.10. In the sequel we will omit the superscript $\mathscr{L}$ in our notations whenever the filtration $\mathscr{L}$ is understood.

Definition 2.11. Let L be a Lie algebra and let $\mathscr{L}$ be an N-series of L. A derivation $t : L \to M$ into an L-module M is called $\mathscr{L}$-layerwise nonsingular if the map $\mathrm{Gr}(t)$ is injective.

Note that the canonical embedding $u : L \to \overline{\mathscr{U}}(L)$ is an L-derivation, if we consider $\overline{\mathscr{U}}(L)$ as an L-module *via left multiplication.* It is now worthwhile mentioning that $F_{u,i}(\overline{\mathscr{U}}(L)) = F_i(L)$. Observe also that $F^{\gamma}_{t,i}(M) = L^{i-1} \cdot M$, and so we rediscover the usual filtration of an L-module. In particular, we find that $F^{\gamma}_i(L) = \overline{\mathscr{U}}(L)^i$.

The following lemma, the proof of which is left to the reader, is rather trivial, but nevertheless crucial to what follows.

Lemma 2.12. *Let L be a Lie algebra and suppose M and Q are L-modules. If $t : L \to M$ is an L-derivation and $q : M \to Q$ is an L-linear map, then*
(1) $qt : L \to Q$ is an L-derivation and
(2) if q is surjective, $F_{qt,i}(Q) = qF_{t,i}(M)$, for all $i \geq 1$.

The importance of the derivation $u : L \to \overline{\mathscr{U}}(L)$ is now made clear:

Proposition 2.13. *The canonical map $u : L \to \overline{\mathscr{U}}(L)$ is the universal derivation of L into L-modules, i.e. for any L-module M and any derivation $t : L \to M$ there exists a unique L-linear map $t' : \overline{\mathscr{U}}(L) \to M$ such that $t'u = t$. Moreover, u is an $\mathscr{L}$-layerwise nonsingular derivation.*

Proof. The fact that u is a universal derivation can be found in [8, p. 234], the essential point to show is that u is $\mathscr{L}$–layerwise nonsingular. For all $i \geq 1$ we choose a set of elements $x_{i,j} \in \mathscr{L}_i$ $(1 \leq j \leq k_i)$, in such a way that the set $\{x_{i,j} + \mathscr{L}_{i+1} \parallel 1 \leq j \leq k_i\}$ forms a basis of $L_i = \mathscr{L}_i / \mathscr{L}_{i+1}$. If $\mathscr{L}$ is of finite length, the total collection of $x_{i,j}$'s will form a basis of L, if not, we add elements $x_{\infty,j}$ until we obtain a basis. We order this basis as follows:

$$x_{p,q} < x_{r,s} \quad \text{iff } p < r \text{ or } (p = r \text{ and } q < s).$$

By the Poincaré–Birkhoff–Witt theorem we know that the ordered monomials

$$x_{i_1,j_1} x_{i_2,j_2} \dots x_{i_s,j_s}, \quad s \in \mathbb{N}, \; x_{i_1,j_1} \leq x_{i_2,j_2} \leq \dots \leq x_{i_s,j_s}$$

form a basis of $\mathscr{U}(L)$. If we define the weight w of such a basis vector by

$$w(1) = 0, \qquad w(x_{i_1,j_1} x_{i_2,j_2} \dots x_{i_s,j_s}) = \sum_{k=1}^{s} i_k,$$

one easily checks, using the defining relations in $\overline{\mathscr{U}}(L)$, that $F_i(L)$ is spanned by those basis vectors of weight $\geq i$. This implies that the set

$$B_i = \{v + F_{i+1}(L) \parallel v \text{ is a basis vector of weight} = i\}$$

forms a basis for $F_i(L)/F_{i+1}(L) = \mathrm{Gr}_i(\overline{\mathscr{U}}(L))$. But the map $\mathrm{Gr}_i(u): \mathrm{Gr}_i(L) \to \mathrm{Gr}_i(\overline{\mathscr{U}}(L))$ maps the basis $\{x_{i,j} + \mathscr{L}_{i+1} \parallel 1 \leq j \leq k_i\}$ onto the subset $\{x_{i,j} + F_{i+1} \parallel 1 \leq j \leq k_i\}$ of B_i. So we conclude that u is an $\mathscr{L}$-layerwise nonsingular derivation. □

Lemma 2.14. *Suppose $t: L \to M$ is a derivation of L into an L-module M, such that M is generated, as an L-module, by* $\mathrm{Im}(t)$. *Then*

$$F_{t,i}(M) = t' F_i(L), \quad \forall i \geq 1,$$

where t' is the map from Proposition 2.13.

Proof. This is an immediate consequence of Lemma 2.12, since the condition that M is generated by $\mathrm{Im}(t)$ implies that $t': \overline{\mathscr{U}}(L) \to M$ is an epimorphism. □

Layerwise nonsingular derivations are useful in the context of nilpotent Lie algebras thanks to the following elementary lemma:

Lemma 2.15. *Let L be a Lie algebra equipped with an N-series of finite length. Then an $\mathscr{L}$-layerwise nonsingular derivation is nonsingular and is a k-isomorphism if it is minimal.*

The converse is not true, even in case $\mathscr{L} = \gamma$, i.e. there exist nonsingular and minimal derivations $t: L \to M$, for which $\mathrm{Gr}(t)$ is not injective.

Example 2.16. Let L be the abelian Lie algebra k^2. Define an L-module structure on $M = k^2$ via

$$\lambda: L \to \mathrm{gl}(M): \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \mapsto \begin{pmatrix} 0 & q_2 \\ 0 & 0 \end{pmatrix}.$$

It is easy to see that the identity map $t = 1_{k^2}: k^2 = L \to k^2 = M$ is an L-derivation. For the filtration $\mathscr{L} = \gamma$ of length one, one checks that the filtration on M is given by

$$M \supseteq L \cdot M = \left\{ \begin{pmatrix} q \\ 0 \end{pmatrix} \parallel q \in k \right\} \supseteq L^2 \cdot M = 0.$$

This implies that the map $\mathrm{Gr}(t)$ is not faithful, although t itself was an isomorphism.

As a generalization of nilpotent modules, we introduce the notion of an $(\mathscr{L}, t)$-nilpotent module.

Definition 2.17. An $(\mathscr{L}, t)$-nilpotent module consists of an L-module M and an L-derivation $t: L \to M$ such that $F^{\mathscr{L}}_{t,n}(M) = 0$ for some n.

If n is such that $F^{\mathscr{L}}_{t,n+1}(M) = 0$, we say that M is n-step $(\mathscr{L}, t)$-nilpotent.

Remark 2.18.
- A module which is $(\mathscr{L}, t)$-nilpotent is nilpotent in the usual sense.
- A module is (γ, t) nilpotent if and only if it is nilpotent.

In order to construct the universal example of an n-step $(\mathscr{L}, t)$-nilpotent module, we truncate the module $\overline{\mathscr{U}}(L)$. For $n \geq 1$ we define the "polynomial construction"

$$P_n^{\mathscr{L}}(L) = \overline{\mathscr{U}}(L)/F_{n+1}^{\mathscr{L}}(L) = F_1^{\mathscr{L}}(L)/F_{n+1}^{\mathscr{L}}(L),$$

and we let $u_n : L \to P_n^{\mathscr{L}}(L)$ be equal to u followed by the quotient map.

To justify this notation and terminology, we include the following

Remark 2.19. Suppose that $k = \mathbb{R}$. Consider the Lie group G_L of a nilpotent Lie algebra L which is defined by taking $G_L = L$ as a set and where the multiplication is given by the Campbell–Baker–Hausdorff formula. Suppose further that L has rational structure constants; then G_L has a lattice subgroup, say G. Now let $\mathscr{L}$ be an N-series of L. Putting $\mathscr{N}_i = \mathscr{L}_i \cap G$ this induces an N-series of G [13, p. 20], $G = \mathscr{N}_1 \supset \mathscr{N}_2 \supset \cdots$, which gives rise to a filtration $\mathbb{R}(G) = A_o \supset A_1 \supset \cdots$ of $\mathbb{R}(G)$ by two-sided ideals [13, p. 22]. Then there is an isomorphism of $\mathbb{R}$-algebras (without unit)

$$P_n^{\mathscr{L}}(L) \overset{\phi}{\cong} A_1/A_{n+1}$$

such that the following square commutes:

$$\begin{array}{ccc} P_n^{\mathscr{L}}(L) & \overset{\phi}{\cong} & A_1/A_{n+1} \\ \uparrow{\scriptstyle u_n} & & \uparrow{\scriptstyle p_n} \\ L = G_L & \hookleftarrow & G \end{array}$$

with $p_n(a) = a - 1 + A_{n+1}$. In particular, for $\mathscr{L} = \gamma$ we find an $\mathbb{R}$-algebra isomorphism with the classical polynomial construction for groups introduced by Passi in [12],

$$P_n^{\gamma}(L) \overset{\phi}{\cong} P_{n,\mathbb{R}}(G).$$

Notation 2.20. Again, we remark that we will often drop the superscript $\mathscr{L}$ in our notation for the polynomial construction.

We get the following analogue of Proposition 2.13:

Proposition 2.21. *The canonical map $u_n : L \to P_n(L)$ is the universal derivation of L into n-step $(\mathscr{L}, t)$-nilpotent L-modules, i.e. for any derivation $t : L \to M$ into an n-step $(\mathscr{L}, t)$-nilpotent L-module, there exists a unique L-linear map $t' : P_n(L) \to M$ such that $t'u = t$. Moreover, if $\mathscr{L}$ is of length $\leq n$, u_n is an $\mathscr{L}$-layerwise nonsingular derivation.*

The following lemma shows the connection between affine representations of strong canonical form and minimal $\mathscr{L}$-layerwise nonsingular derivations:

Lemma 2.22. *Let L be a Lie algebra of dimension m and $\mathscr{L}$ be any N-series of L of length n, then the following are equivalent:*

(1) L admits an affine representation $\rho: L \to \mathfrak{aff}(m)$, which is of strong canonical form with respect to $\mathscr{L}$.

(2) L admits a minimal $\mathscr{L}$-layerwise nonsingular derivation $t: L \to M$ into an n-step $(\mathscr{L}, t)$-nilpotent module M.

Proof. Suppose $\rho: L \to \mathfrak{aff}(m)$ is an affine representation which is of strong canonical form with respect to the N-series $\mathscr{L}$. As discussed before, we can consider $\mathrm{tr}(\rho)$ as being an L-derivation, with respect to the L-module structure of k^m induced by $\mathrm{lin}(\rho)$. Recall the subspaces $M_i = \mathrm{tr}(\rho)(\mathscr{L}_i)$ required in the definition of a strong canonical form affine representation. Remark that, using the notations and the results of Lemma 2.14, we find that

$$\begin{aligned} F_{t,i}(k^m) &= t' F_i(L) \\ &= t' \left(\sum_{\substack{s \\ i_1 + i_2 + \cdots + i_s \geq i}} \mathscr{L}_{i_1} \mathscr{L}_{i_2} \ldots \mathscr{L}_{i_s} \right) \\ &= \sum_{\substack{s \\ i_1 + i_2 + \cdots + i_s \geq i}} \mathscr{L}_{i_1} \mathscr{L}_{i_2} \cdots \mathscr{L}_{i_{s-1}} t(\mathscr{L}_{i_s}) \\ &\subseteq M_i. \end{aligned}$$

The other inclusion $M_i \subseteq F_{t,i}(k^m)$ is trivial. We conclude that k^m is an n-step $(\mathscr{L}, t)$-nilpotent module and that t is a minimal $\mathscr{L}$-layerwise nonsingular derivation.

Conversely, assume $t: L \to M$ is a minimal $\mathscr{L}$-layerwise nonsingular derivation into an n-step $(\mathscr{L}, t)$-nilpotent module M. Without loss of generality, we may assume that $M = k^m$ and we define $K_i = t(\mathscr{L}_i)$ $(1 \leq i \leq n)$ and $K_i = 0$ $(i > n)$. Write the module structure of k^m by means of the map $\lambda: L \to \mathrm{gl}(k^m)$. We claim that the map

$$\rho: L \to \mathfrak{aff}(m): x \mapsto \begin{pmatrix} \lambda(x) & t(x) \\ 0 & 0 \end{pmatrix}$$

is an affine representation which is in strong canonical form. The only thing left to show is that

$$\mathscr{L}_i K_j \subseteq K_{i+j}, \quad \forall i, j. \tag{1}$$

We first prove, by descending induction on n that $F_{t,i}(k^m) = K_i$. For $i = n + 1$, the result is immediate since $F_{t,n+1}(k^m) = 0 = K_{n+1}$. So suppose the result holds for all values $> i$. Using the fact that t is minimal $\mathscr{L}$-layerwise nonsingular we know that

$\mathrm{Gr}_i(t): L_i \to F_{t,i}(k^m)/F_{t,i+1}(k^m)$ is an isomorphism. So

$$F_{t,i}(k^m)/F_{t,i+1}(k^m) = t(\mathscr{L}_i)/K_{i+1} = K_i/K_{i+1}$$

from which it follows that $F_{t,i}(k^m) = K_i$.

Now we see that $\mathscr{L}_i K_j = t'(\mathscr{L}_i \mathscr{L}_j) \subseteq t'(F_{i+j}(L)) = F_{t,i+j}(k^m) = K_{i+j}$, which proves (1). □

3. Layerwise complementary submodules

The aim of this section is to provide another translation of Milnor's problem. If K is a submodule of $P_n(L)$, we let $K_i = K \cap F_i$, where $F_i = F_i(L)/F_{n+1}(L)$, $1 \leq i \leq n$, denote the induced filtration of K. $\mathrm{Gr}(K)$ is the associated graded $\mathrm{Gr}(L)$-module.

Definition 3.1. K is called $\mathscr{L}$-layerwise complementary if

$$F_i/F_{i+1} = \mathrm{Gr}_i(u_n)(L_i) \oplus K_i/K_{i+1}, \quad \forall i \in \{1,2,\ldots,n\}$$

Remark 3.2.

(1) To see K_i/K_{i+1} as a subspace of F_i/F_{i+1} we use the canonical isomorphism $K_i/K_{i+1} \cong (K_i + F_{i+1}(L))/F_{i+1}(L)$.

(2) If K is $\mathscr{L}$-layerwise complementary, then $P_n(L) = u_n(L) \oplus K$ as k-spaces.

Let $t: L \to M$ be a derivation into an L-module such that M is n-step $(\mathscr{L}, t)$-nilpotent. Let M' be the L-submodule of M generated by $\mathrm{Im}(t)$ and let $t'': L \to M'$ be the derivation defined by $t''(x) = t(x)$. Consider the L-linear map $t': P_n(L) \to M'$ which satisfies $t'u_n = t''$, see Proposition 2.21.

Proposition 3.3. *With the notations introduced above and assuming $\mathscr{L}$ is of length $\leq n$, we have the following equivalence:*

The derivation $t'': L \to M'$ is minimal and $\mathscr{L}$-layerwise nonsingular iff $K = \mathrm{Ker}(t')$ *is an $\mathscr{L}$-layerwise complementary submodule.*

Proof. We first show that for all i

$$K_i/K_{i+1} = \mathrm{Ker}(\mathrm{Gr}_i(t')). \tag{2}$$

The inclusion $K_i/K_{i+1} \subseteq \mathrm{Ker}(\mathrm{Gr}_i(t'))$ is obvious. So let $x \in F_i(L)$ such that $\mathrm{Gr}_i(t')(x + F_{i+1}(L)) = 0$. Then $t'(x) \in F_{t'',i+1}(M') \overset{\text{Lemma 2.14}}{=} t'F_{i+1}(L)$, so $x \in K_i \bmod F_{i+1}(L)$, which proves (2).

Consider the following commutative diagram the row of which is exact by (2):

$$\begin{tikzcd} K_i/K_{i+1} \arrow[r, hook] & F_i/F_{i+1} \arrow[r, "\mathrm{Gr}_i(t')"] & \mathrm{Gr}_{t,i}(M') \\ & L_i \arrow[u, "\mathrm{Gr}_i(u_n)"] \arrow[ur, "\mathrm{Gr}_i(t'')"'] & \end{tikzcd}$$

Now if t'' is minimal and $\mathscr{L}$-layerwise nonsingular then $\mathrm{Gr}_i(t'')$ is injective and surjective (dimension counting) for all i, so $\mathrm{Gr}_i(t')$ is surjective and the section $\mathrm{Gr}_i(u_n)$ $\mathrm{Gr}_i(t'')^{-1}$ provides the desired decomposition

$$F_i/F_{i+1} = \mathrm{Gr}_i(u_n)(L_i) \oplus K_i/K_{i+1} \tag{3}$$

which shows that K is a complementary submodule.

Conversely, suppose that (3) holds for all i. Then $\mathrm{Gr}_i(u_n)(L_i) \cap \mathrm{Ker}(\mathrm{Gr}_i(t')) = 0$ by (2), so $\mathrm{Gr}_i(t'') = \mathrm{Gr}_i(t')\mathrm{Gr}_i(u_n)$ is injective by the $\mathscr{L}$-layerwise nonsingularity of u_n, see Proposition 2.21. Thus t'' is $\mathscr{L}$-layerwise nonsingular. It remains to show that $\mathrm{Gr}(t'')$ is surjective. But for all i,

$$\begin{aligned}\mathrm{Gr}_{t,i}(M') &\overset{\text{Lemma 2.12}}{=} \mathrm{Gr}_i(t')(F_i(L)/F_{i+1}(L)) \\ &\overset{(3),(2)}{=} \mathrm{Gr}_i(t'')(L_i).\end{aligned}$$

This proves the proposition. □

Corollary 3.4. *Let L be a Lie algebra with a finite N-series of length n. Then L admits a minimal $\mathscr{L}$-layerwise nonsingular derivation iff $P_n(L)$ has an $\mathscr{L}$-layerwise complementary submodule.*

In order to construct an $\mathscr{L}$-layerwise complementary submodule K of $P_n(L)$ we first exhibit a canonical candidate for its associated graded module $\mathrm{Gr}(K)$ as follows.

Lemma 3.5. *Let L be a Lie algebra (over k) with an N-series $\mathscr{L}$. Then a k-linear retraction*

$$r_i : \mathrm{Gr}_i(\mathscr{U}(L)) \to L_i = \mathscr{L}_i/\mathscr{L}_{i+1}$$

of the morphism $\mathrm{Gr}_i(u)$ is defined as follows: Let $x_j \in \mathscr{L}_{m_j}$, $1 \le j \le s$ such that $m_1 + \cdots + m_s = i$. Then

$$r_i(x_1 \cdots x_s + F_{i+1}(L)) = \frac{m_s}{m_1 + m_2 + \cdots + m_s}[x_1,[\ldots[x_{s-1},x_s]\ldots]].$$

Proof. First we show that there is a canonical isomorphism of graded algebras:

$$\mathrm{Gr}(\overline{\mathscr{U}}(L)) \cong \overline{\mathscr{U}}(\mathrm{Gr}(L)). \tag{4}$$

To obtain this isomorphism, consider the universal L-derivation $u : L \to \overline{\mathscr{U}}(L)$, producing the $\mathrm{Gr}(L)$-derivation $\mathrm{Gr}(u) : \mathrm{Gr}(L) \to \mathrm{Gr}(\overline{\mathscr{U}}(L))$. Since this $\mathrm{Gr}(L)$-derivation has to factor through the universal one, we obtain a map $\overline{\mathscr{U}}(\mathrm{Gr}(L)) \to \mathrm{Gr}(\overline{\mathscr{U}}(L))$, which can be seen to be an isomorphism using the Poincaré–Birkhoff–Witt theorem. (See also the basis arguments in the proof of Proposition 2.13.)

To continue the proof of the lemma, we endow the graded Lie algebra $\mathrm{Gr}(L)$ with the following action on itself, see [9]: For $x_j \in \mathscr{L}_{m_j}$ write $\bar{x}_j = x_j + \mathscr{L}_{m_j+1}$. Then let

$$\bar{x}_1 \cdot \bar{x}_2 = \frac{m_2}{m_1 + m_2}[x_1, x_2] + \mathscr{L}_{m_1+m_2+1}.$$

The identity map of $\mathrm{Gr}(L)$ is a derivation with respect to this action. Let $R:\overline{\mathscr{U}}(\mathrm{Gr}(L)) \to \mathrm{Gr}(L)$ be its $\mathrm{Gr}(L)$-linear extension, see Proposition 2.13. This is a graded k-linear retraction of the map $u_{\mathrm{Gr}(L)}$. Combining R with the inverse of the canonical isomorphism $\overline{\mathscr{U}}(\mathrm{Gr}(L)) \to \mathrm{Gr}(\overline{\mathscr{U}}(L))$, we get a canonical graded k-linear retraction $R':\mathrm{Gr}(\overline{\mathscr{U}}(L)) \to \mathrm{Gr}(L)$ of $\mathrm{Gr}(u)$. One easily verifies that

$$R_i'(x_1 \cdots x_s + F_{i+1}(L)) = R_i(\bar{x}_1 \cdots \bar{x}_s) = \frac{m_s}{m_1 + \cdots + m_s}[x_1, \ldots, [x_{s-1}, x_s] \ldots] + \mathscr{L}_{i+1}.$$

Thus $r_i = R_i'$ is well defined and has the desired properties. $\square$

Defining $Q = \bigoplus_{i \geq 1} \mathrm{Ker}(r_i)$ we obtain the following

Corollary 3.6. *Let L be a Lie algebra with an N-series $\mathscr{L}$. Then there exists a natural graded $\mathrm{Gr}(L)$-submodule Q of $\mathrm{Gr}(\overline{\mathscr{U}}(L))$ such that for all $i \geq 1$,*

$$F_i(L)/F_{i+1}(L) = \mathrm{Gr}_i(u)(L_i) \oplus Q_i.$$

The goal of the following sections is to lift this module back to an $\mathscr{L}$-layerwise complementary submodule of $P_n(L)$ itself.

4. The 3-step case

Some of the basic information used to construct layerwise complementary submodules is provided by the following two natural short exact sequences involving $\mathrm{S}^i L_1$, the i-fold symmetric tensor power of L_1.

Lemma 4.1. *Let L be any Lie algebra and $\mathscr{L}$ an N-series of L. Then there are natural short exact sequences*

$$L_2 \overset{\mathrm{Gr}_2(u)}{\rightarrowtail} \mathscr{U}_2(\mathrm{Gr}(L)) \twoheadrightarrow \mathrm{S}^2 L_1, \tag{5}$$

$$L_3 \oplus L_1 \otimes L_2 \overset{\alpha}{\rightarrowtail} \mathscr{U}_3(\mathrm{Gr}(L)) \twoheadrightarrow \mathrm{S}^3 L_1, \tag{6}$$

with $\alpha = (\mathrm{Gr}_3(u), \mu(\mathrm{Gr}_1(u) \otimes \mathrm{Gr}_2(u)))$, μ being multiplication in $\mathscr{U}(\mathrm{Gr}(L))$.

Proof. The projection p of $\mathrm{Gr}(L)$ onto $L_1 = L/\mathscr{L}_2 = \mathscr{L}_1/\mathscr{L}_2$ defines canonically a graded algebra map $\tilde{p} : \mathscr{U}(\mathrm{Gr}(L)) \to \mathscr{U}(L_1) = \mathrm{S}L_1$, the symmetric algebra of L_1. Since p is surjective $\tilde{p}$ is surjective and $I = \mathrm{Ker}(\tilde{p})$ is the ideal generated by $u(\mathrm{Ker}(p)) \subseteq \mathscr{U}(\mathrm{Gr}(L))$ (e.g. see [3]). Since $\mathrm{Ker}(p)$ is generated by elements of homogeneous degree, we obtain short exact sequences

$$I_i \rightarrowtail \mathscr{U}_i(\mathrm{Gr}(L)) \twoheadrightarrow \mathrm{S}^i L_1.$$

Using the Poincaré–Birkhoff–Witt theorem, one checks that the degree 2 and 3 parts are given as claimed above. $\square$

Notation 4.2. For the rest of this paper, let us fix elements $x_{1,j} \in L$ $(1 \le j \le k_1)$ such that the set $\{\bar{x}_{1,j} = x_{1,j} + \mathscr{L}_2 \parallel j \in \{1,2,\dots,k_1\}\}$ is a basis of L_1.

Definition 4.3.

(1) σ_i is the k–linear map

$$\sigma_i : \mathrm{S}^i L_1 \to P_n(L) \colon \sigma_i(\bar{x}_{1,j_1} \hat{\otimes} \cdots \hat{\otimes} \bar{x}_{1,j_i}) = \frac{1}{i\,!} \sum_{\sigma \in \Sigma_i} u_n(x_{1,\sigma(j_1)}) \cdots u_n(x_{1,\sigma(j_i)}).$$

(2) $i_n : \mathrm{Gr}_n(\mathscr{U}(L)) \rightarrowtail P_n(L)$ denotes the canonical inclusion.

Theorem 4.4. *The subspace*

$$K = \mathrm{Im}(\sigma_2) + i_3(\mathrm{Ker}(r_3))$$

is an $\mathscr{L}$-layerwise complementary submodule of $P_3(L)$.

Proof. K is a submodule of $P_3(L)$ since $L \cdot K = L \cdot \mathrm{Im}(\sigma_2) \subseteq i_3(\mathrm{Ker}(r_3)) \subseteq K$. Consider the following exact sequence of K–homomorphisms:

$$\begin{aligned} \mathrm{S}^2 L_1 \overset{\sigma_2}{\twoheadrightarrow} K_2/K_3 \rightarrowtail F_2/F_3 &\twoheadrightarrow (F_2/F_3)/\mathrm{Gr}_2(u)(L_2) \\ &\cong \mathscr{U}_2\mathrm{Gr}(L)/\mathrm{Gr}_2(u)(L_2) \\ &\cong \mathrm{S}^2 L_1 \end{aligned}$$

The penultimate isomorphism is given by (4) and the last one by (5). The composite map of the above sequence is the identity map, so we conclude that σ_2 and the composite map $K_2/K_3 \to (F_2/F_3)/\mathrm{Gr}_2(u)(L_2)$ are isomorphisms. This implies that

$$F_2/F_3 = \mathrm{Gr}_2(u)(L_2) \oplus K_2/K_3 \tag{7}$$

and that

$$\mathrm{Im}(\sigma_2) \cap F_3 = 0. \tag{8}$$

Thus $K_3 = i_3(\mathrm{Ker}(r_3))$ and $F_3/F_4 = \mathrm{Gr}_3(u)(L_3) \oplus K_3$ by Lemma 3.5. □

Combined with Corollory 3.4 and Lemma 2.22 this theorem implies that all 3-step nilpotent Lie algebras admit a strong canonical form affine representation. At this point, it seems useful to point out that with this method, we do find exactly the same representations as Scheuneman did in [14]. Although Scheuneman only uses the lower central series, his proofs are also valid for any N-series of length 3. In our notations, his result may be stated as follows:

Let L be any Lie algebra with an N-series

$$L = \mathscr{L}_1 \supseteq \mathscr{L}_2 \supseteq \mathscr{L}_3 \supseteq \mathscr{L}_4 = 0.$$

Let $\{x_{i,j} \parallel i \in \{1,2,3\},\ j \in \{1,2,\ldots,k_i\}\}$ be a basis of L such that $x_{i,j} \in \mathscr{L}_i - \mathscr{L}_{i+1}$ ($i = 1,2$ or 3). If the Lie brackets in L are given by

$$[x_{1,i},x_{1,j}] = \sum_k a_{i,k,j}x_{2,k} + \sum_k c_{i,k,j}x_{3,k},$$
$$[x_{1,i},x_{2,j}] = \sum_k b_{i,k,j}x_{3,k},$$

then L admits a faithful affine representation (in fact, in strong canonical form) determined by

$$x_{1,i} \mapsto \begin{pmatrix} 0 & \frac{2}{3}B_i & \frac{1}{2}C_i & 0 \\ 0 & 0 & \frac{1}{2}A_i & 0 \\ 0 & 0 & 0 & X_{1,i} \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad x_{2,i} \mapsto \begin{pmatrix} 0 & 0 & \frac{1}{3}D_i & 0 \\ 0 & 0 & 0 & X_{2,i} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad x_{3,i} \mapsto \begin{pmatrix} 0 & 0 & 0 & X_{3,i} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $X_{i,j}$ denotes the k_i-column vector with a 1 on the jth spot and zeroes elsewhere and where

$$(A_i)_{k,j} = a_{i,k,j}, \quad (C_i)_{j,k} = c_{i,j,k}, \quad (B_i)_{j,k} = b_{i,j,k} \text{ and } (D_i)_{j,k} = -b_{j,k,i}.$$

In order to see that these results coincide with ours, we have to check that the kernel of the map t' associated with the representation defined above via Proposition 2.21, coincides with the module K introduced in Theorem 4.4. By the fact that K and $\mathrm{Ker}(t')$ (by Proposition 3.3 and Lemma 2.22) are both $\mathscr{L}$-layerwise complementary submodules, it suffices to prove that $K \subseteq \mathrm{Ker}(t')$.

(1) $\mathrm{Im}(\sigma_2) \subseteq \mathrm{Ker}(t')$, since

$$\begin{aligned} t'(2\sigma_2(\bar{x}_{1,i}\hat{\otimes}\bar{x}_{1,j})) &= t'(x_{1,i}x_{1,j} + x_{1,j}x_{1,i}) = x_{1,i}t(x_{1,j}) + x_{1,j}t(x_{1,i}) \\ &= \begin{pmatrix} 0 & \frac{2}{3}B_i & \frac{1}{2}C_i \\ 0 & 0 & \frac{1}{2}A_i \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ X_{1,j} \end{pmatrix} + \begin{pmatrix} 0 & \frac{2}{3}B_j & \frac{1}{2}C_j \\ 0 & 0 & \frac{1}{2}A_j \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ X_{1,i} \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} \sum_k c_{i,k,j}X_{3,k} + \sum_k c_{j,k,i}X_{3,k} \\ \sum_k a_{i,k,j}X_{2,k} + \sum_k a_{j,k,i}X_{2,k} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

(2) When we restrict everything to $F_3(L)$, we see that the maps t' and r_3 are essentially the same. This can be checked by computing the values of both maps on the basis elements of $F_3(L)$. Here we only make the calculations for elements of the form $x_{1,i}x_{2,j}$.

$$\begin{aligned} t'(x_{1,i}x_{2,j}) &= \begin{pmatrix} 0 & \frac{2}{3}B_i & \frac{1}{2}C_i \\ 0 & 0 & \frac{1}{2}A_i \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ X_{2,j} \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \sum_k b_{i,k,j}X_{3,k} \\ 0 \\ 0 \end{pmatrix}, \end{aligned}$$

$$r_3(x_{1,i}x_{2,j}) = \frac{2}{3}[x_{1,i},x_{2,j}]$$
$$= \frac{2}{3}\sum_k b_{i,k,j}x_{3,k},$$

From this it follows at once that $\mathrm{Ker}(r_3) \subseteq \mathrm{Ker}(t')$.

Remark 4.5. We remark that our method has the advantage above Scheuneman's, that we do not need to keep track of the relations between the structure constants. These are embedded implicitly in our work. This advantage becomes even more clear when considering 4-step nilpotent Lie algebras. We suspect that the difficulty of controlling these relations is also the reason why it took such a long time before new results concerning the affine representations of nilpotent Lie algebras could be presented. These difficulties were also pointed out on the group level in [5].

5. The 4-step case

As usual, we denote by $\Lambda^2 L_1$ the exterior tensor square of L_1 over k. Our main result, Theorem A, can now be reformulated and proved as follows:

Theorem 5.1. *Let L be a Lie algebra and $\mathscr{L}$ an N-series of L of length 4. If the canonical map*

$$\Lambda^2 L_1 \to L_2$$

induced by the bracket in L is injective, then $P_4(L)$ admits an $\mathscr{L}$-layerwise complementary submodule. This implies that L admits an affine representation of strong canonical form (see Corollary 3.4 and Lemma 2.22).

Proof. As before we fix elements $x_{1,j} \in L$ such that $A = \{\bar{x}_{1,j} = x_{1,j} + \mathscr{L}_2 \parallel 1 \le j \le k_1\}$ forms a basis of L_1. By the hypothesis of the theorem the elements of $C = \{\overline{[x_{1,i},x_{1,j}]} = [x_{1,i},x_{1,j}] + \mathscr{L}_3 \parallel 1 \le i < j \le k_1\}$ are linearly independent and so we can extend this set C by elements $y_{2,1},\ldots,y_{2,k} \in \mathscr{L}_2$ such that $B = C \cup \{y_{2,1} + \mathscr{L}_3,\ldots,y_{2,k} + \mathscr{L}_3\}$ forms a basis of L_2. Having fixed such basises A and B we define a k-linear homomorphism $l: L_1 \otimes L_2 \to P_4(L)$ by

$$\forall \bar{x}_{1,i} \in A, \forall \bar{x}_{2,j} \in B : l(\bar{x}_{1,i} \otimes \bar{x}_{2,j}) = x_{1,i}x_{2,j} - \tfrac{2}{3}[x_{1,i},x_{2,j}],$$

where, from now onwards, we suppress the u_n from the notations. Define subspaces of $P_4(L)$ by

$$K^2 = \mathrm{Im}(\sigma_2), \quad K^3 = \mathrm{Im}(l) + \mathrm{Im}(\sigma_3), \quad K^4 = i_4(\mathrm{Ker}(r_4)), \qquad K = K^2 + K^3 + K^4.$$

To check that K is a submodule of $P_4(L)$, observe that $L \cdot K^4 = 0$, $L \cdot K^3 \subseteq K^4$ and $\mathscr{L}_2 K^2 \subseteq K^4$ by Lemma 4.1. For $x,y,z \in \{x_{1,j} \parallel j \in \{1,2,\ldots,k_1\}\}$ one has

$$z(xy + yx) \overset{(*)}{=} l(\bar{y} \otimes \overline{[z,x]} + \bar{x} \otimes \overline{[z,y]}) + 2\sigma_3(\bar{x}\hat{\otimes}\bar{y}\hat{\otimes}\bar{z}) \in K^3,$$

whence $x_{1,j}K^2 \subseteq K^3$ for all $1 \le j \le k_1$. At this point we make crucial use of the hypothesis which ensured that we could prescribe the values of l (and thus K^3) in such a way that equation $(*)$ holds.

Having proved that K is a submodule, we now check that it is $\mathscr{L}$-layerwise complementary. For degree 2 we recall (7) from the proof of the 3-step case. For degree 3 and 4 consider the following sequence of k-homomorphisms:

$$L_1 \otimes L_2 \oplus S^3 L_1 \overset{(l,\sigma_3)}{\twoheadrightarrow} K_3/K_4 \rightarrowtail F_3/F_4 \twoheadrightarrow (F_3/F_4)/\mathrm{Gr}_3(u)(L_3).$$

The map (l, σ_3) is surjective by (8). Now the composite map of the above sequence is an isomorphism by the identification $F_3/F_4 \cong \mathscr{U}_3(\mathrm{Gr}(L))$, see (4), and by (6). As in the proof of Theorem 4.4 we conclude that

$$F_3/F_4 = \mathrm{Gr}_3(u)(L_3) \oplus K_3/K_4 \quad \text{and} \quad K_3 \cap F_4 = 0.$$

This implies $K_4 = K^4$ and thus $F_4/F_5 = \mathrm{Gr}_4(u)(L_4) \oplus K_4$ by Lemma 3.5, as required. □

Proof of Theorem B. For the given Lie algebra L we can define an N-series $\mathscr{L}$ of length 4 by

$$\mathscr{L}_1 = L, \quad \mathscr{L}_2 = U, \quad \mathscr{L}_3 = [L, U], \quad \mathscr{L}_4 = [U, U] + [L, [L, U]] \text{ and } \mathscr{L}_5 = 0.$$

The conditions of the theorem imply that $\mathscr{L}$ satisfies all the criteria for an N-series and that the canonical map $\Lambda^2 L_1 \to L_2$ is injective. Therefore, the preceeding theorem can be applied. □

Proof of Theorem C. Define $U = \mathrm{Span}(S \cup [L, L])$ and apply Theorem A. □

Finally, we want to indicate how to obtain the matrix representation for the Lie algebras satisfying the conditions of Theorem 5.1. Choose a basis $\{x_{i,j} \parallel i \in \{1,2,3,4\},\ j \in \{1,2,\ldots,k_i\}\}$ of L, such that $x_{i,j} \in \mathscr{L}_i \backslash \mathscr{L}_{i+1}$. Moreover, choose the $x_{2,j}$'s as explained in the proof of Theorem 5.1. Recall also the definition of the $\mathscr{L}$-layerwise complementary submodule K of $P_4(L)$. The set $\{x_{i,j} + K \parallel i \in \{1,2,3,4\},\ j \in \{1,2,\ldots,k_i\}\}$ will form a basis of the module $P_4(L)/K$. So we easily find an isomorphism between $P_4(L)/K$ and L. After this identification we see that the composite map

$$L \xrightarrow{u_n} P_4(L) \to P_4(L)/K \cong L$$

is the identity map. This map is the derivation (i.e. the translational part of the affine representation), we were looking for. Therefore, we only need a description of the module structure (i.e. the linear part) of $P_4(L)/K$. Let us make the following computations in $P_4(L)$:

$$x_{1,i}x_{4,j} = 0 \Rightarrow x_{1,j} \cdot (x_{4,j} + K) = 0,$$
$$x_{1,i}x_{3,j} = \underbrace{\tfrac{1}{4}x_{1,i}x_{3,j} + \tfrac{3}{4}x_{3,j}x_{1,i}}_{\in \mathrm{Ker}(r_4)} + \tfrac{3}{4}[x_{1,i}, x_{3,j}] \Rightarrow x_{1,i} \cdot (x_{3,i} + K) = \tfrac{3}{4}[x_{1,i}, x_{3,j}] + K,$$

$$x_{1,i}x_{2,j} = l(\bar{x}_{1,i} \otimes \bar{x}_{2,j}) + \tfrac{2}{3}[x_{1,i}, x_{2,j}] \Rightarrow x_{1,i} \cdot (x_{2,j} + K) = \tfrac{2}{3}[x_{1,i}, x_{2,j}] + K,$$
$$x_{1,i}x_{1,j} = \sigma_2(\bar{x}_{1,i} \hat{\otimes} \bar{x}_{1,j}) + \tfrac{1}{2}[x_{1,i}, x_{1,j}] \Rightarrow x_{1,i} \cdot (x_{1,j} + K) = \tfrac{1}{2}[x_{1,i}, x_{1,j}] + K,$$

$$x_{2,i}x_{4,j} = 0 \Rightarrow x_{2,i} \cdot (x_{4,j} + K) = 0,$$
$$x_{2,i}x_{3,j} = 0 \Rightarrow x_{2,i} \cdot (x_{3,j} + K) = 0,$$
$$x_{2,i}x_{2,j} = \underbrace{\tfrac{1}{2}(x_{2,i}x_{2,j} + x_{2,j}x_{2,i})}_{\in \mathrm{Ker}(r_4)} + \tfrac{1}{2}[x_{2,i}, x_{2,j}] \Rightarrow x_{2,i} \cdot (x_{2,j} + K) = \tfrac{1}{2}[x_{2,i}, x_{2,j}] + K,$$
$$x_{2,i}x_{1,j} = l(\bar{x}_{1,j} \otimes \bar{x}_{2,i}) + \tfrac{1}{3}[x_{2,i}, x_{1,j}] \Rightarrow x_{2,i} \cdot (x_{1,j} + K) = \tfrac{1}{3}[x_{2,i}, x_{1,j}] + K,$$

$$x_{3,i}x_{4,j} = 0, \Rightarrow x_{3,i} \cdot (x_{4,j} + K) = 0,$$
$$x_{3,i}x_{3,j} = 0 \Rightarrow x_{3,i} \cdot (x_{3,j} + K) = 0,$$
$$x_{3,i}x_{2,j} = 0 \Rightarrow x_{3,i} \cdot (x_{2,j} + K) = 0,$$
$$x_{3,i}x_{1,j} = \underbrace{\tfrac{3}{4}x_{3,i}x_{1,j} + \tfrac{1}{4}x_{1,j}x_{3,i}}_{\in \mathrm{Ker}(r_4)} + \tfrac{1}{4}[x_{3,i}, x_{1,j}] \Rightarrow x_{3,i} \cdot (x_{1,j} + K) = \tfrac{1}{4}[x_{3,i}, x_{1,j}] + K,$$
$$x_{4,i}x_{4,j} = 0 \Rightarrow x_{4,i} \cdot (x_{4,j} + K) = 0,$$
$$x_{4,i}x_{3,j} = 0 \Rightarrow x_{4,i} \cdot (x_{3,j} + K) = 0,$$
$$x_{4,i}x_{2,j} = 0 \Rightarrow x_{4,i} \cdot (x_{2,j} + K) = 0,$$
$$x_{4,i}x_{1,j} = 0 \Rightarrow x_{4,i} \cdot (x_{1,j} + K) = 0.$$

This implies that if we order the basis of L as follows:

$$x_{4,1}, x_{4,2}, \ldots, x_{4,k_4}, x_{3,1}, \ldots, x_{3,k_3}, x_{2,1}, \ldots, x_{1,k_1}$$

and if we use the symbol $X_{i,j}$ to denote the k_i column vector with a one on the jth spot and zeroes elsewhere, we find the following relation between the matrices of the adjoint representation and those of the obtained affine representation ρ:

$$\text{If } \mathrm{ad}(x_{1,i}) = \begin{pmatrix} O_{k_4} & A & B & C \\ 0 & O_{k_3} & D & E \\ 0 & 0 & O_{k_2} & F \\ 0 & 0 & 0 & O_{k_1} \end{pmatrix} \quad \text{then} \quad \rho(x_{1,i}) = \begin{pmatrix} O_{k_4} & \frac{3}{4}A & \frac{2}{3}B & \frac{1}{2}C & 0 \\ 0 & O_{k_3} & \frac{2}{3}D & \frac{1}{2}E & 0 \\ 0 & 0 & O_{k_2} & \frac{1}{2}F & 0 \\ 0 & 0 & 0 & O_{k_1} & X_{1,k_1} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$\text{If } \mathrm{ad}(x_{2,i}) = \begin{pmatrix} O_{k_4} & 0 & G & H \\ 0 & O_{k_3} & 0 & I \\ 0 & 0 & O_{k_2} & 0 \\ 0 & 0 & 0 & O_{k_1} \end{pmatrix} \quad \text{then} \quad \rho(x_{2,i}) = \begin{pmatrix} O_{k_4} & 0 & \frac{1}{2}G & \frac{1}{3}H & 0 \\ 0 & O_{k_3} & 0 & \frac{1}{3}I & 0 \\ 0 & 0 & O_{k_2} & 0 & X_{2,i} \\ 0 & 0 & 0 & O_{k_1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$\text{If } \mathrm{ad}(x_{3,i}) = \begin{pmatrix} O_{k_4} & 0 & 0 & J \\ 0 & O_{k_3} & 0 & 0 \\ 0 & 0 & O_{k_2} & 0 \\ 0 & 0 & 0 & O_{k_1} \end{pmatrix} \quad \text{then} \quad \rho(x_{3,i}) = \begin{pmatrix} O_{k_4} & 0 & 0 & \frac{1}{4}J & 0 \\ 0 & O_{k_3} & 0 & 0 & X_{3,i} \\ 0 & 0 & O_{k_2} & 0 & 0 \\ 0 & 0 & 0 & O_{k_1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$\text{If } \mathrm{ad}(x_{4,i}) = \begin{pmatrix} O_{k_4} & 0 & 0 & 0 \\ 0 & O_{k_3} & 0 & 0 \\ 0 & 0 & O_{k_2} & 0 \\ 0 & 0 & 0 & O_{k_1} \end{pmatrix} \quad \text{then} \quad \rho(x_{4,i}) = \begin{pmatrix} O_{k_4} & 0 & 0 & 0 & X_{4,i} \\ 0 & O_{k_3} & 0 & 0 & 0 \\ 0 & 0 & O_{k_2} & 0 & 0 \\ 0 & 0 & 0 & O_{k_1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

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